Numericals based on Hermitian operator

1. ⇒ Examine if d^2/dx^2 is a Hermitian operator.

Sol^n ⇒ for an operator to satisfy the Hermitian condition we say that if an operator $\hat{A}$ has two eigenfunctions ψ and ϕ and if

$$\int \psi(\hat{A}\phi)d\tau = \int(\hat{A}\psi)\cdot\phi d\tau \quad (\psi \text{ and } \phi \text{ are real})$$

or,
$$\int \psi^*(\hat{A}\phi)d\tau = \int(\hat{A}\psi)^*\cdot\phi d\tau \quad (\psi \text{ and } \phi \text{ are Complex})$$

Then $\hat{A}$ is called the Hermitian operator.

Let, $\psi = e^{ix}$ and $\phi = \sin x$ be the two acceptable eigenfunctions. Then,

$$\int \psi(\hat{A}\phi)d\tau = \int e^{-ix}\frac{d}{dx^2}(\sin x)\,dx = -\int e^{-ix}\sin x\,dx$$

$$\int \phi(\hat{A}\psi)^* d\tau = \int \sin x\left[\frac{d^2}{dx^2}(e^{ix})\right]^* dx = \int \sin x\,(i^2 e^{ix})^* dx$$

$$= -\int \sin x\, e^{-ix}\,dx$$

Since the two integrals are the same, $\frac{d^2}{dx^2}$ is Hermitian.

2. ⇒ Show that the operator $\hat{P}_x$ for Linear momentum is Hermitian.

Sol^n ⇒ It is required to prove that

$$\int_{-\infty}^{\infty}\psi^*\left(-i\hbar\frac{d}{dx}\right)\phi\,dx = \int_{-\infty}^{\infty}\phi\left(-i\hbar\frac{d}{dx}\right)^*\psi^*\,dx$$

Integrating by Parts, the left hand Side is equal to

$$-i\hbar\left[\varphi^*\phi\right]_{-\infty}^{\infty} - (i\hbar)\int_{-\infty}^{\infty}\phi\,\frac{d\varphi^*}{dx}\,dx = 0 + \int_{-\infty}^{\infty}\phi\left(i\hbar\frac{d}{dx}\right)\varphi^*\,dx$$

$$= \int_{-\infty}^{\infty}\phi\left(-i\hbar\frac{d}{dx}\right)^*\varphi^*\,dx = R.H.S.$$

Hence $\hat{p}_x$ is a Hermitian operator.

3. ⇒ Show that if two operators $\hat{A}$ and $\hat{B}$ are Hermitian, then their product $(\hat{A}\hat{B})$ is also Hermitian if and only if $\hat{A}$ and $\hat{B}$ commute.

Soln ⇒

Since $\hat{A}\hat{B} = \hat{B}\hat{A}$ (given)

$$\int\varphi^*(\hat{A}\hat{B})\,\phi\,d\tau = \int\varphi^*(\hat{B}\hat{A})\,\phi\,d\tau = \int\varphi^*\hat{B}(\hat{A}\phi)\,d\tau$$

Since $\hat{B}$ is Hermitian, the integral is equal to

$$\int(\hat{A}\phi)(\hat{B}\varphi)^*d\tau = \int(\hat{B}\varphi)^*(\hat{A}\phi)\,d\tau = \int\phi\,\hat{A}^*(B\varphi)^*\,d\tau$$

$$= \int\phi(\hat{A}\hat{B})^*\,\varphi^*\,d\tau \quad (\because \hat{A} \text{ is also Hermitian})$$

Hence, $\hat{A}\hat{B}$ is also Hermitian.

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