Permutation Groups

Transformation (definition): — Let $X \neq \phi$. Any map $f: X \to X$ is called a transformation i.e. any map from a set onto itself is a transformation of the set.

Permutation (def.) — Let X be a non-empty finite set. A one-one onto map $f: X \to X$ is called a permutation.

The number of elements in the finite set X is known as degree of the permutation.

Example: — Let the permutations f and g on $S = \{1, 2, 3, 4\}$ be given by

$$f = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix}, \quad g = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 2 & 3 \end{pmatrix}$$

then $f(1) = 3$, $f(2) = 4$, $f(3) = 1$, $f(4) = 2$.

$g(1) = 4$, $g(2) = 1$, $g(3) = 2$, $g(4) = 3$.

Equality of two permutations. Let f and g be two permutations on a set X.
Then we define $f = g$ iff $f(x) = g(x)\ \forall x \in X$.

Example: — Let f and g be given by

$$f = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 3 & 1 \end{pmatrix}, \quad g = \begin{pmatrix} 2 & 1 & 4 & 3 \\ 4 & 2 & 1 & 3 \end{pmatrix}$$

Evidently $f(1) = 2$, $f(2) = 4$, $f(3) = 3$, $f(4) = 1$.

$g(2) = 4$, $g(1) = 2$, $g(4) = 1$, $g(3) = 3$.

From which we have

$f(1) = 2 = g(1)$, $f(2) = 4 = g(2)$, $f(3) = 3 = g(3)$

$f(4) = 1 = g(4)$

This $\Rightarrow f(x) = g(x)\ \forall x \in \{1, 2, 3, 4\}$.

$\Rightarrow f = g$.

Even and odd permutations.

Definition: — A permutation is said to be even or odd according as it can be expressed as a product of even or odd number of transpositions.

Example:

(i) $\begin{pmatrix} 1, 2, 3 \\ 1, 2, 3 \end{pmatrix}$ no inversion.

(ii) $\begin{pmatrix} 1, 2, 3 \\ 3, 1, 2 \end{pmatrix}$ two inversions.

(iii) $\begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}$ three inversions.

(iv) $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 3 & 2 & 4 & 1 \end{pmatrix}$ 8 inversions.

(v) $\begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}$ 1 inversion.

hence (i) Permutation is even
(ii) Permutation is even
(iii) Permutation is odd
(iv) Permutation is even
(v) Permutation is odd.

Examples

(1) A transposition is always an odd permutation.

(2) An identity permutation is considered as even permutation.

(3) The product of two even permutations is an even permutation.

(4) The product of two odd permutations is an even permutation.

(5) The product of an even and an odd is an odd permutation.

Definition — The set of $\frac{n!}{2}$ even permutations of degree n is called alternating set and is denoted by A_n.