Cosets.

Definition: Suppose H is a subgroup of group $(G, \cdot)$. let $a \in G$ be arbitrary.

We define $aH = \{ah : h \in H\}$, $Ha = \{ha : h \in H\}$.

evidently $aH \subset G$, $Ha \subset G$.

aH is called left coset of H in G generated by a.

Ha is called right coset of H in G generated by a.

If e is the identity for G, then $e \in H$ is also identity for H.

$$a = ae \in aH, \quad a = ea \in Ha.$$

This proves that any left coset or right coset of H in G is not empty.

observe that $He = H = eH$.

Hence H itself is right as well as left coset.

If the group $(G, \cdot)$ is abelian, then $ah = ha \; \forall h \in H$

so that $aH = Ha \; \forall a \in G$.

It does not mean that we can never find subgroups H of a non-abelian group G such that $aH = Ha$ for any $a \in G$.

Example-1. Determine the coset decomposition of a multiplicative group $G = \{1, -1, i, -i\}$ w.r.t its subgroup $H = \{1, -1\}$.

Solution: $Ha = \{ha : h \in H\}$ and G is commutative and so $Ha = aH$

also $Ha = H \; \forall a \in H$

distinct right coset is $H(-i) = H(i)$,

where $H(i) = \{i, -i\}$ Ans.

Exam: 02 If $H = \{(1), (12), (34), (12)(34)\}$ is a subgroup of S_4 and $a = (1432)$ is an element of S_4, then

$Ha = \{(1), (1432), (12)(1432), (34)$

$(1432), \ldots \ldots)$

after simplyfying, we get

$Ha = \{(1432), (243), (142), (24)\}$

$aH = \{(1432), (143), (132), (13)\}$.

Theorem — Any two right cosets of a subgroup are either disjoint or identical.

Proof: — Let Ha and Hb be two right cosets of a group H in a group G so that $a, b \in G$.

To prove that $Ha \cap Hb = \phi$ or $Ha = Hb$.

There arise two cases—

(i) $Ha \cap Hb = \phi$ (ii) $Ha \cap Hb \neq \phi$

(ii) $\Rightarrow \exists x \in Ha \cap Hb \Rightarrow x \in Ha$ and $x \in Hb$

$\Rightarrow \exists h_1, h_2 \in H$ such that $x = h_1 a, x = h_2 b$

$\Rightarrow h_1 a = h_2 b \Rightarrow a = (h_1^{-1} h_2) b \Rightarrow a = h_3 b$, where $h_3 = h_1^{-1} h_2$.

Since H is a subgroup and therefore

$h_1, h_2 \in H \Rightarrow h_1^{-1} h_2 \in H \Rightarrow h_3 \in H \Rightarrow Hh_3 = H$ —— (1)

Since $a = h_3 b$

Hence $Ha = Hh_3 b = (Hh_3) b = Hb$ —— by (1).

Thus $Ha \cap Hb \neq \phi \Rightarrow Ha = Hb$.

Thus we have prove that

$Ha = Hb$, $Ha \cap Hb = \phi$.

Proved