Lagrange's and Charpit's Methods for Solving Partial Differential Equations

Introduction:

Partial Differential Equations (PDEs) involve partial derivatives of an unknown function of two or more variables. Two important methods for solving first-order PDEs are Lagrange's Method and Charpit's Method. Lagrange's method is used to solve linear first-order PDEs, while Charpit's method is applied to nonlinear first-order PDEs.

1. Lagrange's Method (Linear First-Order PDE):

The standard form of a linear first-order PDE is: $Pp + Qq = R$, where $p = \partial z/\partial x$ and $q = \partial z/\partial y$.

The auxiliary equations are given by: $dx/P = dy/Q = dz/R$.

By solving these simultaneous equations, we obtain two independent integrals: $u(x, y, z) = C1$ and $v(x, y, z) = C2$. The general solution is expressed as $F(u, v) = 0$.

Example:

Solve: $x\,p + y\,q = z$.

Auxiliary equations: $dx/x = dy/y = dz/z$. Integrating: $\ln x = \ln y + C1 \rightarrow x/y = C1$, and $\ln x = \ln z + C2 \rightarrow x/z = C2$. Hence, the solution is $F(x/y, x/z) = 0$.

2. Charpit's Method (Nonlinear First-Order PDE):

For a nonlinear first-order PDE of the form F(x, y, z, p, q) = 0, Charpit's auxiliary equations are:

dx / (-Fp) = dy / (-Fq) = dz / (pFp + qFq) = dp / (Fx + pFz) = dq / (Fy + qFz).

These equations are solved simultaneously to obtain complete integrals. Charpit's method is powerful for handling nonlinear equations where Lagrange's method cannot be applied.

Example:

Solve: p^2 + q^2 = 1.

Here F = p^2 + q^2 - 1 = 0. Using Charpit's equations and simplifying, we get p = a, q = b, where a^2 + b^2 = 1. Integrating, z = ax + by + c, which represents a family of planes.

Conclusion:

Lagrange's method is suitable for solving linear first-order PDEs using auxiliary equations. Charpit's method extends this idea to nonlinear PDEs using a larger system of characteristic equations. Both methods are fundamental techniques in the theory of first-order partial differential equations.